A Naive Algorithm for Feedback Vertex Set

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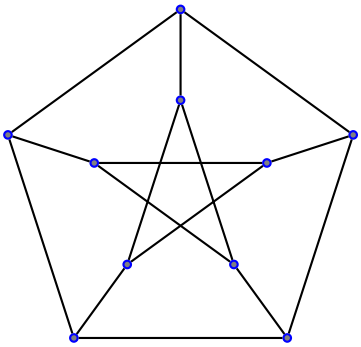
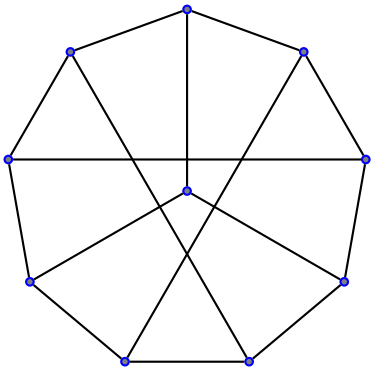
Symposium on Simplicity in Algorithms

New Orleans, LA January 10, 2018

The problem





Given a graph G and an integer k, the feedback vertex set problem asks for the deletion of at most k vertices to make G a forest.

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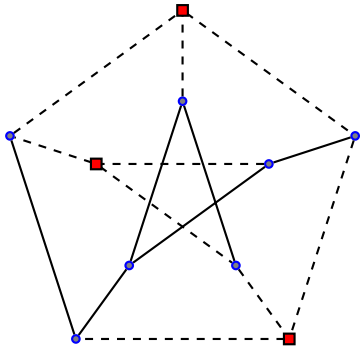

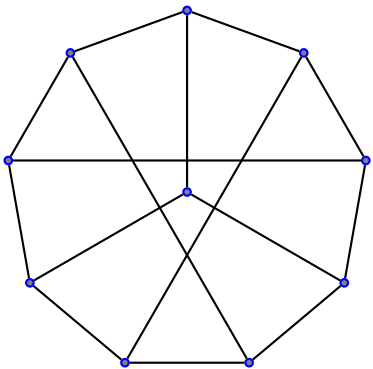

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Origin

The constraint satisfaction problem (CSP) asks for the assignment of values to variables to satisfy a set of constraints.

[Freuder 1982]: can be solved in P-time when the constraint graph is a forest.

[Dechter Pearl 1987]: one way to solve the constraint satisfaction problem is to

1. find first a minimum feedback vertex set V_- of the constraint graph,
2. enumerate all possible assignments on them, and
3. then solve the remaining instance.

On an instance I on p variables, it takes $O(p^{|V_-|} \cdot |I|^{O(1)})$ time.

[Pearl 1988]: A similar application for Bayesian inference.

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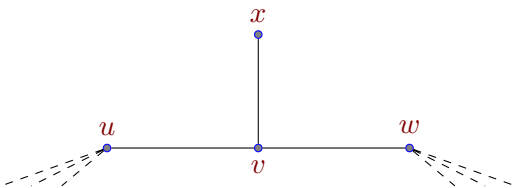
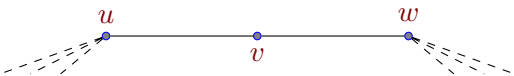
Low-degree vertices



A vertex of degree 1 can always be safely deleted.

Low-degree vertices



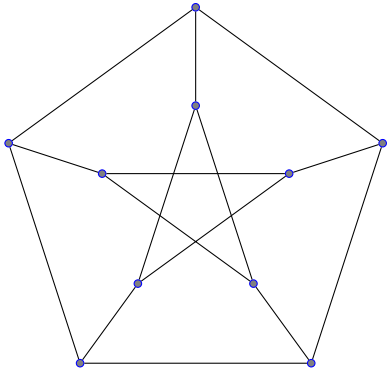
A degree-2 vertex in a solution can always be replaced with one of its neighbors.

Low-degree vertices

If u and w are not adjacent,
then we can delete v and add edge uw.

this operation is called "smoothen."

Low-degree vertices



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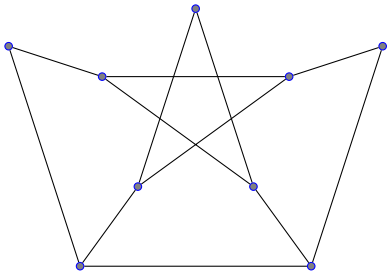

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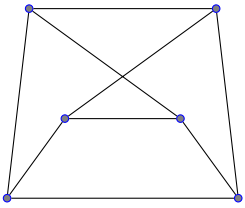

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Greed ... is good. Greed is right. Greed works.

Wall street

Algorithms

[Erdős Pósa 1962]: A graph of minimum degree ≥ 3 has a cycle of length $O(\log n)$.

- A trivial $O(\log n)$ approximation: find a shortest cycle, remove all vertices.
- A nontrivial $(\log k)^{O(k)}$-time parameterized algorithm: after a polynomial kernel.

Large-degree vertices

[Becker Geiger 1996] [Bafna Berman Fujito 1999]: A greedy 2-approximation (Local ratio)

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[Fujito 1996]: Two new primal-dual algorithms

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Parameterized algorithms

Algorithms running in time $O(f(k) \cdot n^{O(1)})$, where k is a parameter (the solution size).

	$f(k)$
Downey Fellows 1992	$(2k+1)^k$
Bodlaender 1994	$17(k^4)!$
Raman et al. 2002	$\max\{12^k, (4\log k)^k\}$
Kanj et al. 2004	$(2\log k + 2\log\log k + 18)^k$
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All c^k algorithms use technique "iterative compression."

Context

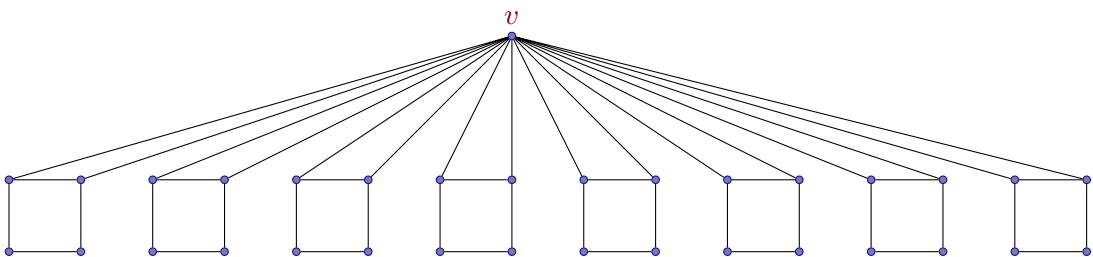
	vertex cover	feedback vertex set
to kill	edges	cycles
to make	independent set (edgeless)	forest (acyclic)
	treewidth 0	treewidth ≤ 1
approx	2	2
parameterized	1.2738^k	3.62^k

the *simplest nontrivial* vertex deletion problem

The Algorithm

A vertex of the largest degree is highly likely in the solution

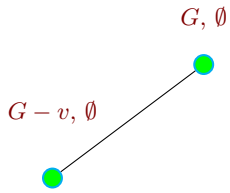
A vertex of the largest degree is highly likely in the solution, but *not* always.



A branching algorithm

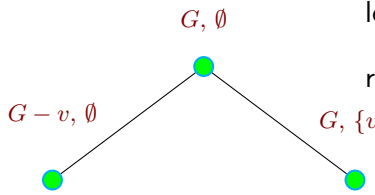
each node has two parts,
the graph G and the set F of *permanent* vertices.

A branching algorithm



left child: the vertex is put into the solution, hence deleted from G

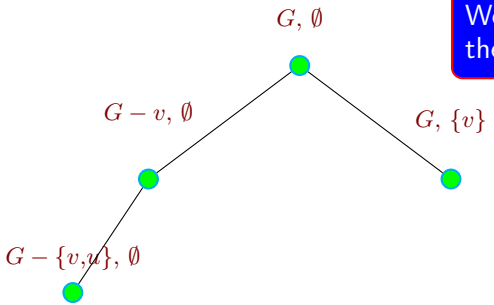
A branching algorithm



left child: the vertex is put into the solution, hence deleted from G

right child: the vertex is not in the solution, hence put into F

A branching algorithm



We always choose the vertex with the largest degree in $V(G) \setminus F$ to branch.

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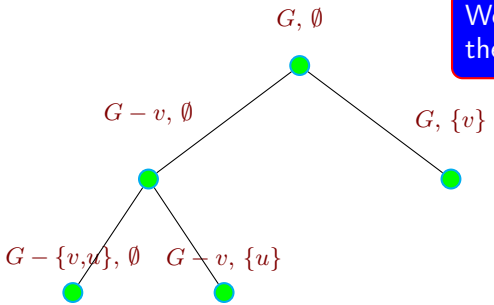

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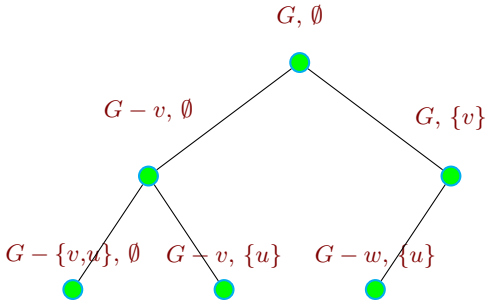

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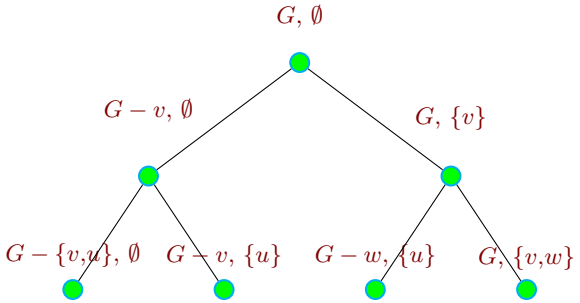

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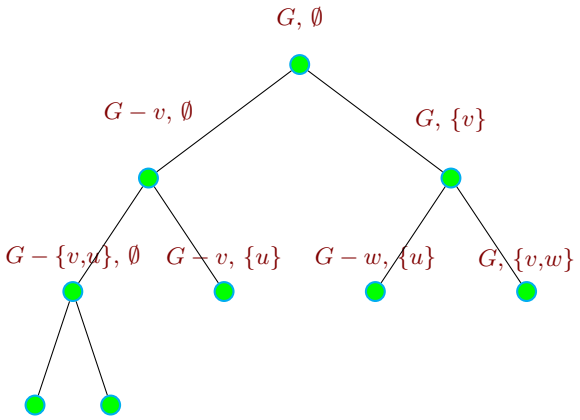

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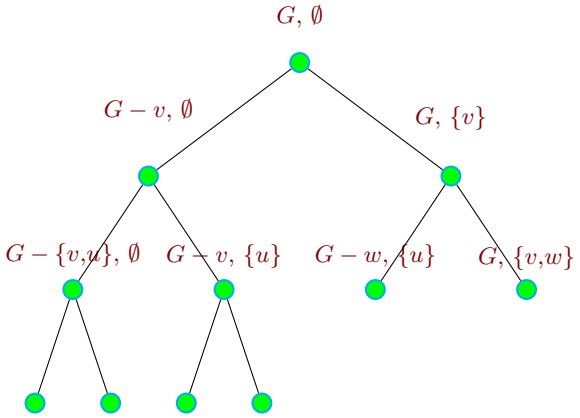

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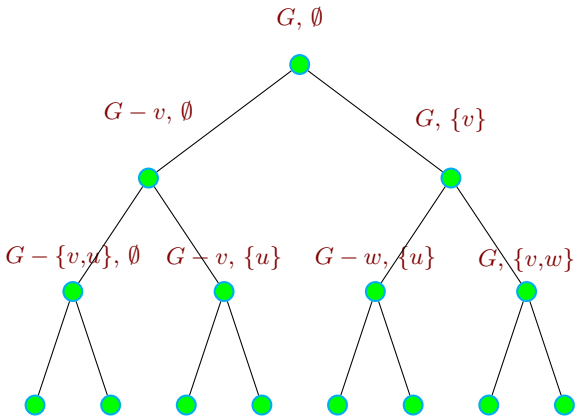
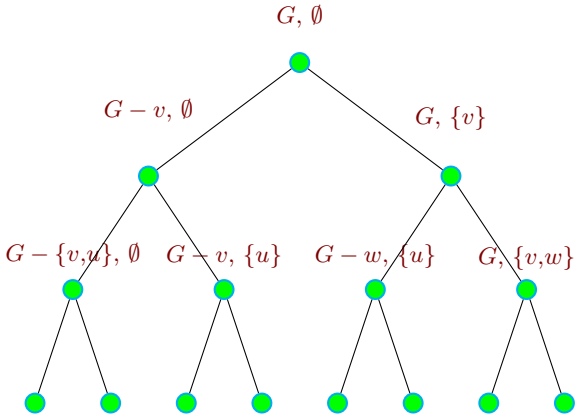


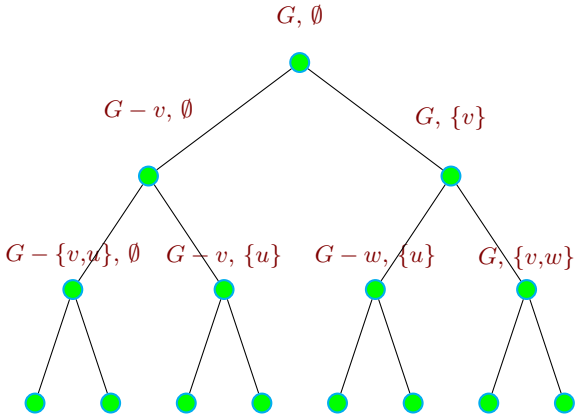
The key of the analysis is to bound the number of nodes, which boils down to bouding the depth.

A branching algorithm



We cannot go left more than k times.
We're hence focused on the right steps.

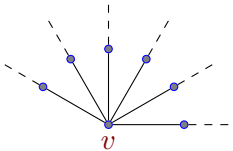
A branching algorithm



Let's fix an execution path.

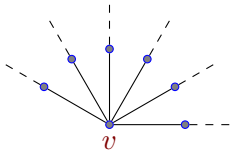
The Analysis

Elementary facts of trees



Elementary facts of trees

$$\sum_{v \in V(T)} d(v) = 2|E(T)| = 2|V(T)| - 2$$



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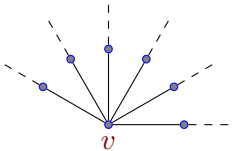

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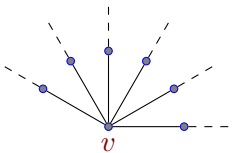

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$$-2 = \sum_{v \in L}(d(v)-2) + \sum_{v \in V(T)\setminus L}(d(v)-2) = \sum_{v \in L}(-1) + \sum_{v \in V_3}(d(v)-2) = \sum_{v \in V_3}(d(v)-2) - |L|.$$

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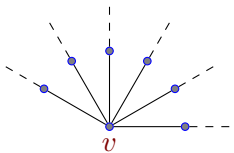

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For each $v \notin V_-$,

- either its degree is decreased from ≥ 3 to ≤ 2,
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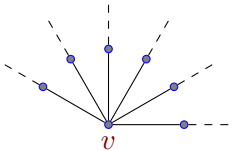

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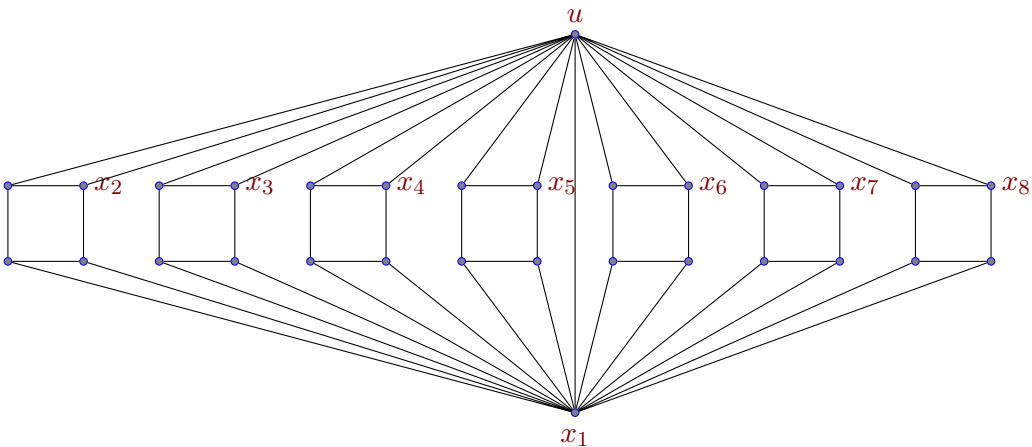
This correlates deleted vertices and permanent vertices.

The example

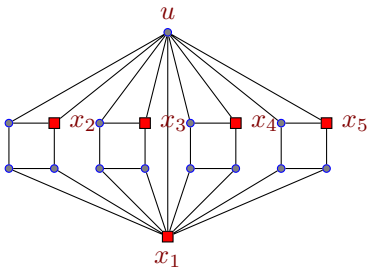
the first vertex x_1 is put into V_-, second u into F,
then the deletion of each x_i decreases its degree by two.



Algorithm invariants

1. During the algorithm, the degree of no vertex can increase.
2. There is no vertex of degree 0 or 1 in the graph when a recursive call is made.

Key definitions and observations



- $d^*(v)$: the degree of v at the moment it is put into V_- or F. $d(v) \geq d^*(v)$
- effective decrements of the degree of a vertex $u \in F$: from $d^*(u)$ to 2.

- an effective decrement is incurred by $x_i \in V_-$ if it is after deleting x_i.
  $\delta(u, x_i)$: #effective decrements of $u \in F$ incurred by $x_i \in V_-$.
  - $\delta(u, x_i)$ may be larger than 1.
  - $\delta(u, x_i) > 0 \quad \Rightarrow \quad u \in F$ when x_i is deleted..

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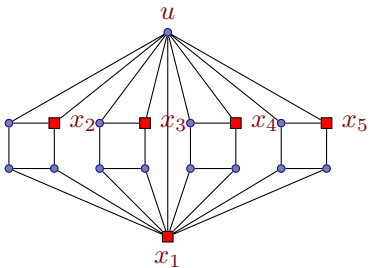

decision points: $x_1 \to u \to x_2 \to x_3 \to x_4 \to x_5$.

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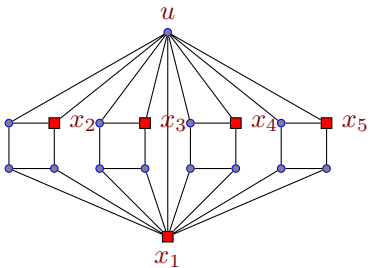

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$d(u) = 9$; $d^*(u) = 8$

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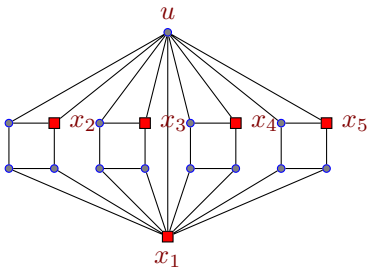

decision points: $x_1 \to u \to x_2 \to x_3 \to x_4 \to x_5$.
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- $d^*(v)$: the degree of v at the moment it is put into V_- or F. $\quad d(v) \geq d^*(v)$
- effective decrements of the degree of a vertex $u \in F$: from $d^*(u)$ to 2.

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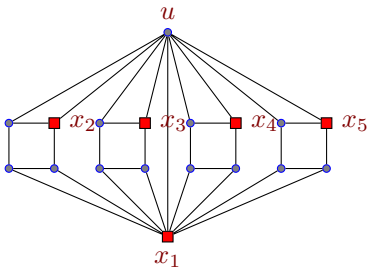

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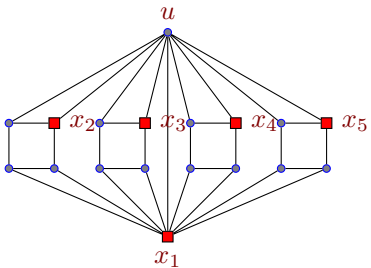

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Lemmas

Lemma 1: For any $u \in F$ and $v \in V_-$, if $\delta(u,v) > 0$ then $d^*(u) \geq d^*(v)$.
Lemma 2: $\sum_{u \in F} \delta(u,v) \leq d^*(v)$ for each $v \in V_-$.

$$|V_-| = \sum_{v \in V_-} 1 = \sum_{v \in V_-} \frac{d^*(v)}{d^*(v)} \geq \sum_{v \in V_-} \frac{1}{d^*(v)} \sum_{u \in F} \delta(u,v)$$

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We terminate all execution paths after $4k$ steps $\Rightarrow O(16^k \cdot n^2)$. □

Epilogue

[Furst Gross McGeoch 1988] feedback vertex set on subcubic graphs is in P.
[C. Chen Liu 2010] This can be extended to the setting $d(v) \leq 3$ for $v \in V(G) \setminus F$.

Only put vertices of degree ≥ 4 into F, then

$$|V_-| \geq \sum_{u \in F} \frac{d^*(u) - 2}{d^*(u)} \geq \sum_{u \in F} \frac{2}{4} = \frac{|F|}{2}.$$

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We terminate all execution paths after $3k$ steps $\Rightarrow O(8^k \cdot n^2)$. $\square$

Final remark

Beauty is the first test:
there is no permanent place in the world for ugly mathematics..

G. H. Hardy

To theorists:
stop ignoring successful heuristic algorithms by pretending their nonexistence!


thanks!